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LETTER TO THE EDITOR

The quantum group $GL_h(2)$

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Abstract. Among all quantum group structures on the space Mat(2) of 2 x 2 matrices, how many have a central quantum determinant, so that one can define quantum SL(2) out of quantum GL(2)? Up to isomorphism, there are two such structures, $GL_q(2)$ and $GL_h(2)$. The former is well known, the latter is described in this paper.

Quantum groups are multiplying like rabbits after rain. Restricting oneself to quantum deformations of the space Mat(n) of $n \times n$ matrices, the variety is still large and growing. In some of these deformations, like $GL_q(n)$, the quantum determinant is central, while in others, like $GL_{p,q}(n)$, it is not. If one imposes the natural requirement that the quantum deformation of GL(n) is restricted to that of SL(n), this amounts to the property of quantum determinant being central. For the case $n = 2$, it is possible to classify *all* such deformations. Up to isomorphism, there exist just two: the well known $GL_q(2)$ (Drinfel'd 1986) and $GL_h(2)$ (or $Mat_h(2)$), given on generators of a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by the relations:

$$\begin{aligned}
ba &= ab + h(a^2 - ad + bc) - h^2ac \\
ca &= ac - hc^2 \\
da &= ad + h(ac - cd) - h^2c^2(1) \\
cb &= bc - h(ac + cd) \\
db &= bd + h(ad - bc - d^2) + h^2ac \\
dc &= cd + hc^2
\end{aligned} \tag{1}$$

where h is a deformation parameter. The quantum determinant

$$\det_h(M) = \mathcal{D} = ad - bc + hac = da - cb - hca \tag{2}$$

is central.

The formulae (1), (2) can be arrived at via the following route. Let us start by classifying all multiplicative Poisson brackets on Mat(2) for which the determinant det is central. It is easy to show that all multiplicative Poisson brackets on Mat(2) result from the condition of being Poisson symmetries of a pair of Poisson planes

$$\{x,y\} = P_2(x,y) \tag{3a}$$

$$\{\xi,\xi\} = Q_2(\xi,\eta) \qquad \{\eta,\eta\} = Q_2'(\xi,\eta) \qquad \{\xi,\eta\} = Q_2''(\xi,\eta) \tag{3b}$$

where $P_2, Q_2, \ldots$ are quadratic polynomials. (A similar property holds for all matrix groups and supergroups.) By a linear change of coordinates, the Poisson bracket (3*a*) can be reduced to one of the two canonical forms:

$$\{x,y\} = xy \tag{4a}$$

and

$$\{x,y\} = y^2 \tag{5a}$$

depending upon whether the roots of P_2 are distinct or not. (If P_2 vanishes then the det is central iff $Q_2 = Q_2' = Q_2'' = 0$.) In these coordinates, the det is central iff the Poisson brackets (3*b*) take the form, respectively,

$$\{\xi,\xi\} = \{\eta,\eta\} = 0 \qquad \{\xi,\eta\} = \eta\xi \tag{4b}$$

and

$$\{\xi,\xi\} = 2\xi\eta \qquad \{\eta,\eta\} = \{\xi,\eta\} = 0. \tag{5b}$$

Upon quantization, the Poisson brackets (4) and (5) become, respectively,

$$xy = q^{-1}yx \qquad \xi^2 = \eta^2 = 0 \qquad \xi\eta = -q\eta\xi \tag{6}$$

and

$$xy = yx + hy^2 \qquad \xi^2 = h\xi\eta \qquad \eta^2 = 0 \qquad \eta\xi = -\xi\eta. \tag{7}$$

The quantum symmetries of (6) lead to the familiar $\mathrm{Mat}_q(2)$ (Manin 1988). The quantum symmetries of (7) result in formulae (1). Formula (2) for the quantum determinant follows from the standard definition

$$\xi'\eta' = \mathcal{D}\xi\eta \qquad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Thus, the quantum determinant is multiplicative. Set

$$\widetilde{M} = \begin{pmatrix} d + hc & -b + h(d - a + hc) \\ -c & a - hc \end{pmatrix}.$$

It is easy to check that

$$\widetilde{M} = \widetilde{M}M = \mathcal{D}\mathbf{1}.$$

From this one deduces by the Takeuchi method that $\mathcal{D}$ is central:

$$\mathcal{D}M = (M\widetilde{M})M = M(\widetilde{M}M) = M\mathcal{D}.$$

Hence,

$$M^{-1} = \mathcal{D}^{-1}\widetilde{M} = \widetilde{M}\mathcal{D}^{-1}.$$

Further properties of $\mathrm{Mat}_h(2)$ follow.

(i) The ring of regular functions on $\mathrm{Mat}_h(2)$ has the PBW property: the monomials $\{a^n b^m c^k d^l \mid n, m, k, l \in \mathbb{Z}_+\}$ form a basis. This is easily checked with the help of the Diamond lemma (Bergman 1978).

(ii) The powers of M are also quantum matrices, with the deformation parameter kh:

$$M^k \in \mathrm{Mat}_{kh}(2) \qquad k \in \mathbb{Z}. \tag{8}$$

This is similar to the other case: $\{M \in \mathrm{Mat}_q(2)\} \Rightarrow \{M^k \in \mathrm{Mat}_{q^k}(2)\}$ (Corrigan *et al* 1990). As in that case,

$$\det_{kh}(M^k) = [\det_h(M)]^k \qquad k \in \mathbb{Z}. \tag{9}$$

(If in formulae (7) one replaces $\xi^2 = h\xi\eta$ by $\xi^2 = -h\xi\eta$, properties (8) and (9) will still hold although $\det_h$ will no longer be central, and these facts remain true for $\mathrm{Mat}_h(n)$.)

(iii) The elements c and a–d are normalizing. Hence, one can self-consistently reduce $\mathrm{Mat}_h(2)$ to the upper-triangular case by setting $c = 0$. Further reductions are possible: $a = d$; $a = d^{-1}$; $a = d$, $b = 0$; $a = d = 1$. Notice that the generators a, c, d form a closed subalgebra.

(iv) The ring of invariants (i.e. the polynomial centre) of $\mathbb{C}[\mathrm{Mat}_h(2)]$ is generated by $\mathcal{D} = \det_h(M)$. The field of invariants is generated by $\mathcal{D}$ and

$$\mathrm{Cas}_2(M) = c^{-1}(a-d) = (a-d)c^{-1}. \tag{10}$$

For this invariant, one has a companion formula to (9):

$$\mathrm{Cas}_2(M^k) = \mathrm{Cas}_2(M) + (k-1)h \qquad 0 \neq k \in \mathbb{Z}. \tag{11}$$

For the case $M \in \mathrm{Mat}_q(2)$ one has similar statements, with (10) being replaced by (Kupershmidt 1991)

$$\mathrm{Cas}_2(M) = c^{-1}b = bc^{-1} \tag{10$'$}$$

and (11) being replaced by

$$\mathrm{Cas}_2(M^k) = q^{1-k}\mathrm{Cas}_2(M) \qquad 0 \neq k \in \mathbb{Z}. \tag{11$'$}$$

(v) Similar to the case of $\mathrm{Mat}_q(2)$ (Ewen *et al* 1991), one has quantum analogues of the Cayley–Hamilton theorem:

$$X^2M^2 = (a+d-hc)XM - \mathcal{D}\mathbf{1} \qquad X = \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \tag{12a}$$

$$M^2Y^2 = MY(a+d+hc) - \mathcal{D}\mathbf{1} \qquad Y = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}. \tag{12b}$$

(vi) If

$$M^k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

we define quantum traces as

$$t_k^\pm = t_k^\pm(M^k) = a_k + d_k \pm khc_k \tag{13}$$

so that

$$t_k^+ = \mathrm{tr}(M^k Y^k) \qquad t_k^- = \mathrm{tr}(X^k M^k).$$

From (12*a*) and (12*b*) we deduce, as in the case of $\mathrm{Mat}_q(2)$ (Kupershmidt 1992), that

$$t_{k+2}^- = t_1^- t_{k+1}^- - \mathcal{D} t_k^- \tag{14a}$$

$$t_{k+2}^+ = t_{k+1}^+ t_1^+ - \mathcal{D} t_k^+. \tag{14b}$$

Since

$$t_{-1}^\pm = \mathcal{D}^{-1} t_1^\pm$$

it follows that all the t_k^+ commute between themselves, $k \in \mathbb{Z}$, and similarly for the t_k^-. Also,

$$ct_k^+ = t_k^- c. \tag{15}$$

Suppose now that

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

is a matrix whose entries commute with those of M. Define

$$\mathrm{Tr}_h(S) = s_{11} + s_{22} - 2hs_{21}. \tag{16}$$

Then

$$\mathrm{Tr}_h(MSM^{-1}) = \mathrm{Tr}_h(S). \tag{17}$$

(vii) In the quasiclassical limit $h \to 0$, formulae (1) yield the following multiplicative Poisson brackets on GL(2):

$$\{a,b\} = ad - bc - a^2 \qquad \{a,c\} = c^2 \qquad \{a,d\} = c(d-a)$$
$$\{b,c\} = c(d+a) \qquad \{b,d\} = d^2 + bc - ad \qquad \{c,d\} = -c^2. \tag{18}$$

Denoting $\overline{f} = \mathrm{d}f|_e$ for any $f \in \mathrm{Fun}(\mathrm{GL}(2))$ (e is the unit element of GL(2)), we can extract from (18) the bi-algebra structure on gl(2), by the formula (Drinfel'd 1986)

$$[\overline{f}, \overline{g}] = \overline{\{f,g\}}.$$

The resulting commutators on $gl(2)^*$ are

$$[\overline{a},\overline{b}] = \overline{d} - \overline{a} \qquad [\overline{a},\overline{c}] = 0 \qquad [\overline{a},\overline{d}] = 0$$
$$[\overline{b},\overline{c}] = 2\overline{c} \qquad [\overline{b},\overline{d}] = \overline{d} - \overline{a} \qquad [\overline{c},\overline{d}] = 0. \tag{19}$$

In the notation

$$I = \overline{a} + \overline{d} \qquad H = \overline{a} - \overline{d} \qquad E = \overline{b} \qquad F = \overline{c}$$

formulae (19) become

$$[E,F] = 2F \qquad [E,H] = 2H \qquad [F,H] = 0 \qquad [I,\ \text{anything}] = 0. \tag{20}$$

The Casimir element is (cf formula (10))

$$\mathrm{Cas} = F^{-1}H = HF^{-1} = \overline{c}^{-1}(\overline{a} - \overline{d}) = (\overline{a} - \overline{d})\overline{c}^{-1}. \tag{21}$$

(viii) Let G be a semisimple Lie group with the Lie algebra $\mathcal{G}$. The ring of regular functions on G, $\mathbb{C}[G]$, consists of matrix elements of finite-dimensional representations $\rho : G \to \mathrm{Aut}(V)$ of G, i.e. of functions $\langle l, \rho(g)(v)\rangle$, $g \in G$, $l \in V^*$, $v \in V$. Let $r \in \Lambda^2(\mathcal{G})$ be a classical r-matrix, so that

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \tag{22}$$

Then the Poisson bracket

$$\{\langle l_1, \rho_1(g)(v_1)\rangle, \langle l_2, \rho_2(g)(v_2)\rangle\}$$
$$= \langle l_1 \otimes l_2, [(\rho_1 \otimes \rho_2)_*(r), \rho_1(g) \otimes \rho_2(g)](v_1 \otimes v_2)\rangle \tag{23}$$

defines a multiplicative Poisson bracket on G. (See Drinfel'd 1986.) When G = GL(n), one can take $\rho_1 = \rho_2$ as the defining representation. Then formula (23) simplifies to

$$\{M_i^\alpha, M_j^\beta\} = \sum_{kl} r^{kl}[(I_k M)_i^\alpha (I_l M)_j^\beta - (M I_k)_i^\alpha (M I_l)_j^\beta] \tag{24}$$

where $r = \sum r^{kl} I_k \otimes I_l$ is the coordinate expression of the r-matrix in a basis of $\mathcal{G}$. For the case at hand, formulae (18) can be written in the form (24) with the r-matrix

$$r = h \otimes e - e \otimes h \tag{25}$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are elements of the basis of [sl(2) and] gl(2). The r-matrix (25) satisfies the classical Yang–Baxter equation (CYBE) (22).

(ix) Many different generalizations are possible for the case of $\mathrm{Mat}(n)$, $n > 2$ (and also for $\mathbb{Z}_2$-graded groups $\mathrm{Mat}(n|m)$). For example, for $n = 3$, apart from obvious versions of formula (25) (depending upon various embeddings of sl(2) into sl(3)), one also has the r-matrix

$$r = -h_1 \wedge (e_1 + 2e_2) + \theta h_2 \wedge (2e_1 + e_2) \qquad \theta \in \mathbb{C} \tag{26}$$

which *does not satisfy* the CYBE (22). (Here $h_i = E_{ii} - E_{ii+1}$, $e_1 = E_{12}$, $e_2 = E_{23}$ in the usual notation.) This situation originates in the symmetries of the three-dimensional Poisson space (Dufour and Haraki 1991)

$$\{x_i, x_{i+1}\} = \partial P / \partial x_{i+2} \qquad i \in \mathbb{Z}_3 \tag{27}$$

where

$$P = x_2^2 x_3 + \theta x_1 x_3^2. \tag{28}$$

In contrast, the r-matrix

$$r = (\alpha_1 h_1 + \alpha_2 h_2) \wedge E_{13} + \epsilon(\alpha_1 + \alpha_2) e_1 \wedge e_2 \qquad \alpha_1, \alpha_2 \in \mathbb{C} \qquad \epsilon = 0, 1 \tag{29}$$

does satisfy the CYBE (22).

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